1/1 (2003), 43–53
tmcs@math.klte.hu
http://tmcs.math.klte.hu

Teaching
Mathematics and
Computer Science

Why is the gamma function so as it is?

Detlef Gronau

Abstract. This is a historical note on the gamma function Γ. The question is, why is $\Gamma(n)$ for naturals n equal to $(n-1)!$ and not equal to $n!$ (the factorial function $n! = 1\cdot 2\cdots n$) ? Was A. M. Legendre responsible for this transformation, or was it L. Euler? And, who was the first who gave a representation of the so called Euler gamma function?

Key words and phrases: history of mathematics, gamma function.

ZDM Subject Classification: 01A55, 39-03, 39B12.

0. Introduction

I often asked myself the following question:

$$\text{“}\textit{Why is } \Gamma(n) = (n-1)! \textit{ and not } \Gamma(n) = n!\ \text{?”} \tag{Q}$$

The standard answer to this question is: Euler[1] introduced the gamma function as an interpolating function for the factorials $n! = \prod_{k=1}^{n} k$. Legendre[2] introduced the notation Γ together with this shift such that $\Gamma(n) = (n-1)!$. But it ain't necessarily so.

As a matter of fact, it was Daniel Bernoulli[3] who gave in 1729 the first representation of an interpolating function of the factorials in form of an infinite product, later known as gamma function.

[1]Leonhard Euler, 15. 4. 1707, Basel – 18.9. 1783, St. Petersburg.
[2]Adrien Marie Legendre, Paris, 18. 9. 1752 – 9. 1. 1833, Paris.
[3]Daniel Bernoulli, 8.2.1700, Groningen – 17. 3. 1782, Basel.

Euler who, at that time, stayed together with D. Bernoulli in St. Petersburg gave a similar representation of this interpolating function. But then, Euler did much more. He gave further representations by integrals, and formulated interesting theorems on the properties of this function.

In this note we give a short sketch on the early history of the gamma function and give a (partial) answer to question (Q).

To begin with let us remind of the generally known representations of the gamma function $\Gamma(x)$, defined for $x \in (0, \infty)$:

$$\Gamma(x) = \lim_{n\to\infty} \frac{n!\ n^x}{x(x+1)\dots(x+n)}, \tag{1}$$

and also

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt, \tag{2}$$

$$\Gamma(x) = \int_0^1 (-\log y)^{x-1} dy, \tag{3}$$

$$\Gamma(x) = \int_{-\infty}^\infty e^{xz} e^{-e^z} dz. \tag{4}$$

All these four representations of the gamma function were given in essential by Euler. The integrals (2), (3) and (4) are equivalent. (3) and (4) follow from (2) by substitution of the variables $y = e^{-t}$ and $z = \log t$, respectively. There is no evidence that Euler gave a proof of the fact that (1) and (2) yield the same function.

1. The interpolation problem

In the 17^{th} century the *problem of interpolation* started to come into fashion (see e.g. [13]). The problem is, for a given function or operation (for example the n^{th} power q^n) which is in a natural way defined for natural numbers n, to find also an expression for non integers, e.g. q^x for all reals x.

Take for example the geometric series and its sum

$$S(n) := \sum_{i=1}^n q^i = 1 + q + \dots + q^n = \frac{q^{(n+1)} - 1}{q - 1}.$$

Here we can express the value of $S(n)$ also for non naturals n, although the definition of $S(n)$ makes sense only for naturals n. More complicated is the case where we don't have an explicit expression for the given function.

Questions of these kind were treated by Christian Goldbach[4] in a comprehensive way during all his life. He considered sums of the form

$$\sum_{i=1}^{n} f(i) \quad \text{with a given function } f : \mathbb{N} \to \mathbb{R}$$

and specially the sum $1 + 1 \cdot 2 + 1 \cdot 2 \cdot 3 + \dots$. The problem is to find the general term of this series.

Goldbach, well known by the *Goldbach conjecture*, has completed a study of law and was an autodidactic mathematician. He published several papers in number theory, infinite series, integration of functions and differential equations. He used to cultivate contacts with several famous mathematicians, so with members of the family Bernoulli and with Leibniz[5]. Goldbach travelled to many places. Finally at the age of 35 in 1725 he earned his first professional position as a secretary of the Academy of Sciences in St. Petersburg. It is not known whether he had contact with Euler already at that time. At the beginning of 1725 Goldbach moved to Moscow as the court of the tsars did. He was tutor of the tsarevitch. From Moscow a correspondence between Euler and Goldbach began which lasted up to the death of Goldbach in 1764 (see [6] and [7]).

The starting point of this extensive correspondence was the later so called *gamma function*. Goldbach, considering especially the problem of interpolating the factorials, asked several mathematicians for advice, so for example in 1722 Nikolaus Bernoulli[6] and later on in 1729 his brother Daniel. In the sequel Goldbach received three letters which were essential for the birth of the gamma function.

2. Three letters on the gamma function

The first letter: The first known letter which contains an interpolating function for the factorials was written by Daniel Bernoulli on October 6, 1729. Bernoulli suggests for an arbitrary (positive) x and an infinite number A the infinite product

$$\left(A + \frac{x}{2}\right)^{x-1} \left(\frac{2}{1+x} \cdot \frac{3}{2+x} \cdot \frac{4}{3+x} \cdots \frac{A}{A-1+x}\right) \tag{5}$$

[4]Christian Goldbach, 18. 3. 1690, Königsberg (now Kaliningrad) – 1. 12. 1764, St. Petersburg.
[5]Gottfried Wilhelm Leibniz, 1. 7. 1646, Leipzig – 14. 11. 1716, Hanover.
[6]Nikolaus II Bernoulli, 6. 2. 1695, Basel – 26. 7. 1726, St. Petersburg.

as interpolating function, hence[7]

$$x! = \lim_{n\to\infty} \left(n+1+\frac{x}{2}\right)^{x-1} \prod_{i=1}^{n} \frac{i+1}{i+x}. \tag{6}$$

For $x = \frac{3}{2}$ and $A = 8$ Bernoulli gets an approximate value $\frac{3}{2}! = 1.3004$ (here Bernoulli fails, I got the value 1.32907) and for $x = 3$ and $A = 16$ the formula (6) yields instead of $3! = 6$ the value $6\frac{1}{204}$. With these remarkable results the correspondence between Bernoulli and Goldbach on interpolation has ended ([6], p. 143).

The second letter: Euler has lived in St. Petersburg since 1727. He gained a position at the Academy of Sciences by the recommendation of Nikolaus and Daniel Bernoulli. So he had good personal contacts to Daniel Bernoulli and knew about the discussion on the interpolating function of the factorials. Euler found his own solution. Encouraged by Bernoulli Euler wrote a letter to Goldbach dated October 13, 1729 containing the *"terminum generalem"* of the argument m:

$$\frac{1\cdot 2^m}{1+m}\cdot\frac{2^{1-m}\cdot 3^m}{2+m}\cdot\frac{3^{1-m}\cdot 4^m}{3+m}\cdot\frac{4^{1-m}\cdot 5^m}{4+m}\ \text{etc.} \tag{7}$$

thus the term of the n^{th} approximation is

$$\frac{1\cdot 2\cdot 3\ldots n\cdot(n+1)^m}{(1+m)(2+m)(3+m)\ldots(n+m)}. \tag{8}$$

This is almost[8] $\Gamma(m+1)$ in the representation (1).

The representation (5) of Bernoulli and that of Euler (7) are formally different, though both formulas (6) and (8) yield in the limit the same value. Numerical experiments show that the formula of Bernoulli converges much faster to its limit than that of Euler. However both of them are equivalent to the product (1) which (in my opinion erroneously) bears the name of Gauss.[9]

The letter of Euler to Goldbach contains much more. So Euler writes that he has calculated for the argument $\frac{1}{2}$ the value of the product (7) as $\frac{\sqrt{\pi}}{2}$. In his own words:

[7]The notation $n!$ was, however, firstly introduced in 1808 by Christian Kramp (1760–1826, professor for mathematics at Strasbourg). Euler used later on in 1771, Eneström No. 421 the notation $[n]$ and in [4] the symbol Δ, i.e. $\Delta(n) = n!$.

[8]The term (8) has the same limit as $\Gamma(m+1)$ in the representation (1).

[9]Carl Friedrich Gauss, 30. 4. 1777, Braunschweig – 23. 2. 1855, Göttingen.

Terminem autem exponentis $\frac{1}{2}$ *aequalis inventus est huic* $\frac{1}{2}\sqrt{(\sqrt{-1}\cdot l-1)}$ *seu, quod huic aequale est, lateri quadrati aequalis circulo, cujus diameter* $= 1$.

This means $\frac{1}{2}! = \frac{1}{2}\sqrt{\sqrt{-1}\cdot\ln(-1)} = \frac{\sqrt{\pi}}{2}$. Euler uses here the extension of the logarithm to negative numbers as he already has discussed with Johann Bernoulli,[10] the father of Nikolaus and Daniel. Here he means the natural logarithm as a multi valued function, especially we get

$$\ln(-1) = i\pi + k\cdot 2\pi i,\ k\in\mathbb{Z} \quad \text{where } i=\sqrt{-1}.$$

In this letter Euler uses for the number π the paraphrase *rationem peripheria ad diametrum*; in later publications, e.g. 1736, he uses also the symbol π, which most probably was first introduced in 1706 by William Jones.[11]

Euler calculates also some special values: $\frac{1}{2}! = 0.8862269$ and $\frac{3}{2}! = \frac{3}{2}\cdot(\frac{1}{2}!) = 1.3293403$. These values coincide with the exact values in all the given decimals. Further he mentions that it is easy to calculate also the values of the factorials for the arguments $\frac{5}{2}$ *etc.* $\frac{1}{4}, \frac{3}{4}, \frac{5}{4}, \frac{7}{4}$ *etc.* and $\frac{1}{8}, \frac{3}{8}, \frac{5}{8}$ *etc.*

THE THIRD LETTER: This is the second letter of Euler to Goldbach, dated January 8, 1730. In this letter Euler first presents an integral representation of the interpolating function of the factorials and explains the properties of a definite integral taken from 0 to 1, where the integrand depends from a further variable. Then he defines the interpolating function for the factorial in the form

$$\int dx(-lx)^n.$$

In our notation this means

$$n! = \int_0^1 (-\ln x)^n dx. \tag{9}$$

Euler writes: *Denotat autem lx logarithmum hyperbolicum ipsius x*. Then, in answering of a question in Goldbach's letter of December 1, 1729, Euler gives a short sketch of the hyperbolic (= natural) logarithm with the aid of an arithmetic and a geometric series where he also emphasizes that this logarithm stems from the quadrature of the hyperbola.

[10]Johann Bernoulli, 6. 8. 1667, Basel – 1. 1. 1748, Basel.

[11]William Jones, 1675, Wales – July 3, 1749, London.

3. The Euler product

How Euler got his infinite product (7)? This can be read in a later paper [4], *De termino generali serium hypergeometricarum.* Here, p. 142, Euler claims: *... quod huiusmodi series in infinitum continuatae tandem cum progressione geometrica confundantur.*[12] So, in Euler's notation, if i is an infinite number then Euler claims (here he uses the term Δ for the interpolating function)

$$\Delta : (i+n) = i^n \Delta : i$$

or, which yields the same

$$\Delta : (i+n) = (i+\alpha)^n \Delta : i$$

for any finite number α. Thus, in our notation, with x instead of n, $n \in \mathbb{N}$ instead of i and $\alpha = 1$ we get from the last formula

$$\Delta(n+x) \approx (n+1)^x \Delta(n) \quad \text{for } n \to \infty.$$

This has to be interpreted as $\lim\limits_{n\to\infty} \frac{(n+1)^x \Delta(n)}{\Delta(n+x)} = 1$, from which one easily gets (8).

4. The Euler integrals

We come back to Euler's second letter to Goldbach and the integral representation of the interpolating function. In his paper [3], 1730, *De progressionibus transcendentibus seu quarum termini generales algebraice dari nequeunt*, Eneström No. 19, Euler treated this problem in more detail. Here, Euler does not use a specific symbol for the interpolating function, but in later papers he uses the symbol Δ. Firstly he considers the above mentioned product (7). He calculates its value for the argument $\frac{1}{2}$. For the sake of convenience we will already here denote the value of the product (7) for the argument m by $\Delta(m)$, hence

$$\Delta(1/2) = \sqrt{\frac{2\cdot 4}{3\cdot 3} \cdot \frac{4\cdot 6}{5\cdot 5} \cdot \frac{6\cdot 8}{7\cdot 7} \cdot \frac{8\cdot 10}{9\cdot 9} \cdot \text{ etc.}} \tag{10}$$

[12]... this series should be determined close to infinity like the geometric series.

Euler compared it with the infinite product representation π given by Wallis.[13] Wallis investigated the integral

$$f(p,n) = \frac{1}{\int_0^1 \left(1 - x^{1/p}\right)^n dx}$$

in connection with the problem of squaring the circle. For naturals n and p we get $f(p,n) = \binom{n+p}{p}$ and for $n = p = \frac{1}{2}$ we have

$$f\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{4}{\pi} = \frac{3}{2} \cdot \frac{3}{4} \cdot \frac{5}{4} \cdot \frac{5}{6} \cdot \frac{7}{6} \cdots .$$

So we have $\Delta(1/2) = \sqrt{\pi}/2$. This showed the genius Euler that integrals are the appropriate tool to give the "right" representation of the interpolating function.

Firstly Euler considered the integral

$$\int x^e dx (1-x)^n, \tag{11}$$

where the integration has to be taken from 0 to 1, hence in our notation

$$E(e,n) = \int_0^1 x^e (1-x)^n dx. \tag{12}$$

This integral, slightly modified was called by Legendre the *Euler integral of the first kind* or *beta function*

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx. \tag{13}$$

It satisfies the identity

$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}. \tag{14}$$

Euler calculates for naturals n

$$(e+n+1) \int x^e dx (1-x)^n = \frac{1 \cdot 2 \cdot 3 \cdots n}{(e+1)(e+2)\dots(e+n)}.$$

Now comes Euler's trick: for e he substitutes $e = \frac{f}{g}$. In this way he gets for naturals n:

$$\frac{1 \cdot 2 \cdot 3 \cdots n}{(f+g)(f+2g)(f+3g)\dots(f+ng)} g^n.$$

[13]John Wallis, 23. 11. 1616, Ashford – 28. 10. 1703, Oxford, professor in Oxford, one of the founding members of the Royal Society.

After some further calculation he comes to

$$\frac{f+(n+1)g}{g^{n+1}}\int x^{\frac{f}{g}}dx(1-x)^n=\frac{1\cdot 2\cdot 3\cdots n}{(f+g)(f+2g)\dots(f+ng)}. \tag{15}$$

The right hand side of (15) yields after substituting $f=1$ und $g=0$ the desired $1\cdot 2\cdots n$. Thus the general term for the factorials is

$$\int\frac{x^{\frac{f}{0}}dx(1-x)^n}{0^{n+1}}.$$

"The meaning of this term will be explained in what follows", writes Euler. Using a well known rule *("regulam igitur cognitam")*, taking the limit $g\to 0$ of the derivatives of both, nominator and denominator inside of the integral, Euler arrives to the expression $\int dx(-lx)^n$. Thus we have another form of the interpolating function of the factorials:

$$\Delta(n)=\int_0^1(-\ln x)^n\ dx. \tag{16}$$

Euler deduced with this from the *Euler integral of the first kind* (11) the (by Legendre) so called *Euler integral of the second kind*.

The here used "well known rule" is nothing else than the rule of de l'Hospital.[14] It goes back to Johann Bernoulli, who has introduced the aristocrat de l'Hospital (for money) into the secrets of higher mathematics. No wonder that Euler was familiar with this rule.

5. Legendre and the gamma function

Adrien Marie Legendre devoted several publications to Euler's integrals. The first time he mentions Euler's first integral in 1792, [10]. Then in 1809, [11] he treats both integrals and introduces the definition of Γ. One can find these presentations in more detail in *Traité des fonctions elliptiques et des integrales Eulériennes,* [12] in 1826. On p. 365 of [12] Legendre writes:

"Quoique le nom d'Euler soit attaché à presque toutes les théories importantes du Calcul intégral, cependant j'ai cru qu'il me serait permis de donner plus spécialement le nom d'*Intégrales Eulériennes*, à deux sortes de transcendantes dont les propriétés ont fait le sujet de plusieurs beaux Mémoires d'Euler,

[14]Guillaume–François-Antoine de l'Hospital, 1661, Paris – 2. 2. 1704, Paris.

et forment la théorie la plus complète que l'on connaisse jusq'à présent sur les intégrales définies.

La première est l'intégrale $\int \frac{x^{p-1}dx}{\sqrt[n]{(1-x^n)^{n-q}}}$ qu'on suppose prise entre les limites $x = 0$, $x = 1$. Nous la représenterons, comme Euler, par la caractère abrégé $(\frac{p}{q})$.

La seconde est l'intégrale $\int dx(\log \frac{1}{x})^{a-1}$, prise de même entre les limites $x = 0$, $x = 1$, que nous représenterons par Γa, et dans laquelle Euler suppose que a est égal à une fraction rationnelle quelconque $\frac{p}{q}$.

Nous considérons ces deux sortes d'intégrales, d'abord sous le même point de vue qu'Euler; ensuite sous un point de vue plus étendu, afin d'en perfectionner la thérie."[15]

Now comes the question: Why introduced Legendre the gamma function in this way that for naturals n we get $\Gamma(n) = (n-1)!$?

Remember, Euler introduced his integral of the second kind with (16), hence $\Delta(a) = \int_0^1 (-\ln x)^a \; dx$. He obtained it from his first integral (11) or (12) such that for naturals n the identity $\Delta(n) = n!$ holds.

The point is that Euler himself made this change in the parameters of his first integral during the time between 1730 and 1768. In Euler's *Institutionem calculi integralis*, Vol. I, caput IX [5], p. 240, Legendre already found the first integral just in this way as he wrote it in his traité, formula (17). If one substitutes in (17) $n = 1$, then one gets exactly the beta function $B(p, q)$ (see (13)). So, Eulers first integral is originally also in this form from Euler.

Legendre deduced from this integral (17) in a similar way like Euler did the second integral in the form $\int_0^1 (\log \frac{1}{x})^{a-1} dx$. This integral is denoted by Legendre

[15]Although the name of Euler could be attached to almost all important theories of the integral calculus, however, I think that it would be permitted to me, specially, to give two types of transcendent functions the name Euler integrals. Their properties have been the subject of several memoirs of Euler and they build the mostly complete theory known about definite integrals.

The first is the integral

$$\int \frac{x^{p-1}dx}{\sqrt[n]{(1-x^n)^{n-q}}}, \tag{17}$$

taken in the limits $x = 0$, $x = 1$. We use like Euler the abbreviation $(\frac{p}{q})$.

The second is the integral $\int dx(\log \frac{1}{x})^{a-1}$, also taken from $x = 0$ to $x = 1$. We will denote it by $\Gamma(a)$, where Euler supposed that a is some rational of the form $\frac{p}{q}$.

We will consider these types of integrals firstly under the same point of view as Euler; after that under a more extended point of view, in order to make the theory more perfect.

with $\Gamma(a)$. From

$$\Gamma(a) = \int_0^1 \left(\log \frac{1}{x}\right)^{a-1} dx$$

he gets easily the recursion formula

$$\Gamma(a+1) = a \cdot \Gamma(a)$$

([11], pp. 476–481, [12], pp. 405–409).

As a conclusion one can say that Euler himself was responsible for the fact that $\Gamma(n) = (n-1)!$ holds. It will be the subject of some more extensive study of the *Opera omnia* to understand why Euler came from [3], 1730 to [5], 1768 to this change of the parameters in his integrals.

One possible reason for the change of the parameter in the first integral could be the following. The nice relation (14) between the beta and gamma function would be in terms of (12) with E and Δ of the form

$$E(m,n) = \frac{\Delta(m) \cdot \Delta(n)}{\Delta(m+n+1)}.$$

This relation is by sure not as nice as (14). But it is hard to believe that this was a motivation for Euler or Legendre to do so as they did.

References

[1] E. Artin, *Einführung in die Theorie der Gammafunktion*, Hamburger Math. Einzelschr., Heft 1, B.G. Teubner, Leipzig/Berlin, 1931, English translation: *The Gamma Function.* Holt, Rinehart and Winston, New York, 1964.

[2] Ph. J. Davis, Leonhard Euler's Integral: A historical Profile of the Gamma function, *Amer. Math. Monthly* **66** (1959), 849–869.

[3] Leonhard Euler, De progressionibus Transcendentibus seu quarum termini generales algebraice dari nequeunt, Eneström, *Opera omnia* seria I, **14**, no. 19, 1730/1, 1–24.

[4] Leonhard Euler, De termino generali serium hypergeometricarum, Eneström, *Opera omnia* seria I, **16a**, no. 652, 1776, 139–162.

[5] Leonhard Euler, *Institutionum calculi integralis*, (volumen primum) Opera omnia, seria I, Petropoli, 1768.

[6] A. P. Juškevič and L. Kh. Kopelevič, *Christian Goldbach, 1690–1764*, Birkhäuser Verlag, Basel etc., 1994.

[7] A. P. Juškevič and E. Winter (Hrsg.), Leonhard Euler und Christian Goldbach, *Briefwechsel 1729–1764*, Abh. der Deutschen Akad. der Wiss. zu Berlin, Akademie-Verlag, Berlin, 1965.

[8] D. Laugwitz and B. Rodewald, Auf Eulers Spuren zur Gammafunktion, *Math. Semesterberichte, Hamburg* **33** (1986), 201–212.

[9] D. Laugwitz and B. Rodewald, A Simple Characterization of the Gamma Function, *Amer. Math. Monthly* **94** (1987), 534–536.

[10] A.-M. Le Gendre, *Mémoires sur les transcendantes elliptiques*, Paris, L'an deuxième de la République, 1792.

[11] M. Legendre, *Recherches sur diverses sortes d'intégrales définies*, Mémoires de la classe des sciences mathématiques et physiques d l'institut de France, Année 1809, 416–509.

[12] A. M. Legendre, *Traité des fonctions elliptiques et des integrales Eulériennes*, Tome second, Paris, 1826.

[13] J. Christoph Scriba, Von Pascals Dreieck zu Eulers Gamma-Funktion, Zur Entwicklung der Methodik der Interpolation, in: *Mathematical Perspectives*, Essays on Mathematics and Its Historical Development, (Joseph W. Dauben, ed.), Academic Press, New York, 1981.

DETLEF GRONAU
INSTITUT FÜR MATHEMATIK
UNIVERSITÄT GRAZ
HEINRICHSTRASSE 36
A-8010 GRAZ
AUSTRIA

E-mail: gronau@uni-graz.at

(Received September 11, 2002)